FORCE

FOCUS ON RESEARCH IN CONTEMPORARY ECONOMICS

GEOMETRIC BROWNIAN MOTION APPROACH TO MODELLING STOCK PRICES

Cigdem Topcu Guloksuz

To cite this article: Topcu Guloksuz, C. (2021). Geometric Brownian motion approach to modelling stock prices. *Focus on Research in Contemporary Economics (FORCE),* *2*(1), 53-63.

To link to this article: https://www.forcejournal.org/index.php/force/article/view/24

Published online: 27 May 2021.

Submission date: 16.03.2021 | Acceptance date: 23.05.2021 RESEARCH ARTICLE

GEOMETRIC BROWNIAN MOTION TO MODELLING STOCK PRICES

Cigdem Topcu Guloksuz*

ABSTRACT

The aim of this study is to revisit the practicability of geometric Brownian motion to modelling of stock prices. Random walk process is extended to the geometric Brownian motion model and its mathematical properties are discussed. The historical data set referring the stock prices of Walmart Company from 16 March 2019 to 13 March 2020 is employed to present the accuracy of the predicted prices, which are generated based on the geometric Brownian motion model. The results display that the geometric Brownian motion model provides accurate predictions.

KEY WORDS: *Random Walk, Brownian Motion, Geometric Brownian Motion, Ito's Lemma, Stock Prices*

*Correspondence concerning this article should be addressed to Cigdem Topcu Guloksuz, Department of Management Information Systems, Ankara Bilim University, Turkey.
E-mail: cigdem.topcu.guloksuz@ankarabilim.edu.tr

1. INTRODUCTION

A stock market refers to a platform which consists of buyers and sellers who trade the financial securities such as stocks/equities, bonds, currencies under set of regulations. When the case is to make profit from the financial activities, modelling the price movements properly becomes an important task to be able get accurate predictions of the prices. The mixed ARMA (p,q)+GARCH(r,s) time series models, the stochastic process can be considered as models for the stock prices. In this study geometric Brownian motion is mainly studied. There are many studies in literature about modelling stock prices with stochastic process, Reddy and Clinton (2016), Almgren (2002), Malliaris (1983). Additionally, several studies in which the geometric Brownian motion is employed as a statistical model of stock prices. For example, Yang (2015) explores some techniques to build financial model using Brownian Motion and Rajpal (2018) considers the Geometric Brownian Motion as a statistical model to predict the Apple's stock price. Azizaha et.al. (2020) compare the performances of the Geometric Brownian Motion and multilayer perceptron for stock price predictions and find that the Geometric Brownian Motion provides more accurate results.

The organization of the paper is as follows: Section 1 introduces the random walk process, Brownian motion and their properties. In Section 2, Geometric Brownian motion is revisited as a stochastic differential equation and the solution of the equation is linked to the predicted future returns. A small application study is conducted in Section 3. The paper is concluded in Conclusion.

2. A BIREF OVERVIEW FOR RANDOM WALK AND BROWNIAN METHOD

A random walk is a path which consists of a set of random steps. The start point is zero and following movement may be one step to the left or to the right with equal probability. In the random walk process, there is no observable trend or pattern which are followed by the objects that is the movements are completely random. That is why the prices of a stock as it moves up and down can be modelled by random a walk process.

Let X_i denotes the i[th] step and takes – 1 and 1 values with equal probability of $\frac{1}{2}$. The start position is set to zero, $W^{(1)}(0)=0$, then the random walk describes the position at time t or after n steps as follows:

$$W^{(1)}(n)=S_n \tag{1}$$

where $S_n=\sum_{i=1}^{n} X_i$.

Here, X is a Bernoulli random variable as follows

$$X_i=\begin{cases}-1, & p=1/2\\ 1, & p=1/2\end{cases} \tag{2}$$

and the expectation and variance of the random walk is

$$E[W^1(t)]=0 \tag{3}$$

$$Var[W^1(t)] = t. \quad (4)$$

Considering the central limit theorem, $n \to \infty$ $W^{(1)}(n) \approx N(0,n)$. It is also note that since one step is taken in one unit time, $W^{(1)}(n) \approx N(0,n)$ and $W^{(1)}(t) \approx N(0,t)$ have same meaning. Figure 1 illustrates an example of a random walk process.

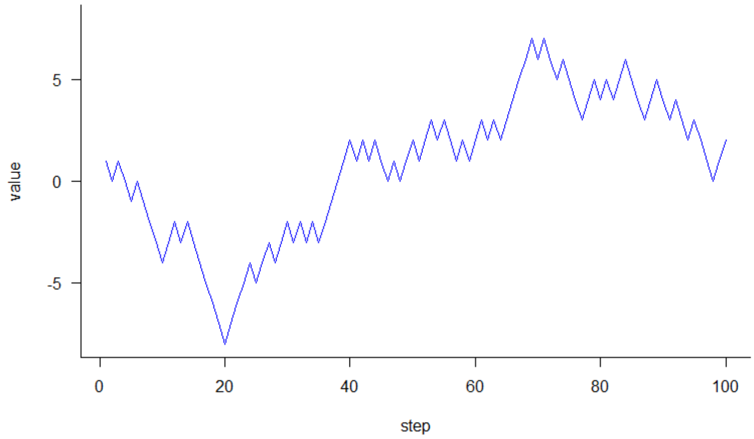

Figure 1: Random Walk with 100 steps in time

For the random walk denoted by $W^1(t)$, it is considered that on step is taken in every unit time. It is possible to consider finer random walk process, represented by $W^k(t)$, in which k steps are taken in each unit of time and the total steps in the random walk is $n = kt$. However, when k grows the variance of the position at time t will be k times larger than the simple random walk process, $W^1(t)$. It means that $W^k(t)$ does not have the desired random walk properties. Recalling the Bernoulli variable in (2), the variance of the $W^k(t)$ is as follows.

$$Var\left[W^{(k)}(t)\right] = Var\left[\sum_{i=1}^{n=kt} X_i\right] = kt \quad (5)$$

To keep the variance constant as in (4), it is needed to rescale the length of step.

$$Y_i = \frac{1}{\sqrt{k}} X_i \tag{6}$$

Then, a general form and the variance of the scaled random walk are obtained in (7) and (8), respectively.

$$W^k(t) = \frac{1}{\sqrt{k}} \sum_{i=1}^{kt} X_i \tag{7}$$

$$Var\left[W^k(t)\right] = Var\left[\frac{1}{\sqrt{k}} \sum_{i=1}^{kt} X_i\right] = t \tag{8}$$

The general properties of the scaled random walk are listed below:

1. Independent increments: for $0 < s < t < l < k$ $W^{(k)}(t) - W^{(k)}(s)$ and $W^{(k)}(k) - W^{(k)}(l)$ are independent.
2. $E[W^{(k)}(t)] = 0$, $Var\left[W^{(k)}(t)\right] = t$
3. $k \to \infty$ $W^{(k)}(t) \approx N(0,t)$.

3. STOCK PRICES AND GEOMETRIC BROWNIAN METHOD

Brownian motion which is firstly realized by the botanist Robert Brown. It is a mathematical model to describe random movements of small particles in a fluid or gas. These random movements are observed in the stock markets where the prices move up and down, randomly. Hence Brownian motion is considered as a mathematical model for stock prices.

A linear function of scaled random walk, $W^k(t)$, with drift, μ, and diffusion coefficient, σ, can defined as follows:

$$B_t^{(k)} = \mu t + \sigma W_t^{(k)} \tag{9}$$

As $k \to \infty$ real valued $\{B(t) : t \geq 0\}$ process is obtained

$$B_t = \mu t + \sigma W_t \tag{10}$$

and has the following properties:

1. For $0 \leq s < t$, the increment, $B(t+s) - B(s) \sim N(0,t)$.
2. B_t has independent increments.
3. B_t is continuous.

B_t is called Brownian motion with drift μ and diffusion coefficient σ. The version with $\mu = 0$ and $\sigma = 1$ is called standard Brownian motion or *Wiener* process and denoted by W_t. It should be noted that W_t can be defined as

$$W_t = \varepsilon\sqrt{t} \tag{11}$$

where $\varepsilon \sim N(0,1)$. The Brownian motion, B_t with drift is normally distributed with following expectation and variance.

$$E(B_t) = E(\mu t + \sigma W_t) = \mu t \tag{12}$$

$$Var(B_t) = Var(\mu t + \sigma W_t) = \sigma^2 t \tag{13}$$

Figure 2 illustrates simulated Brownian motions with positive and negative drift, respectively.

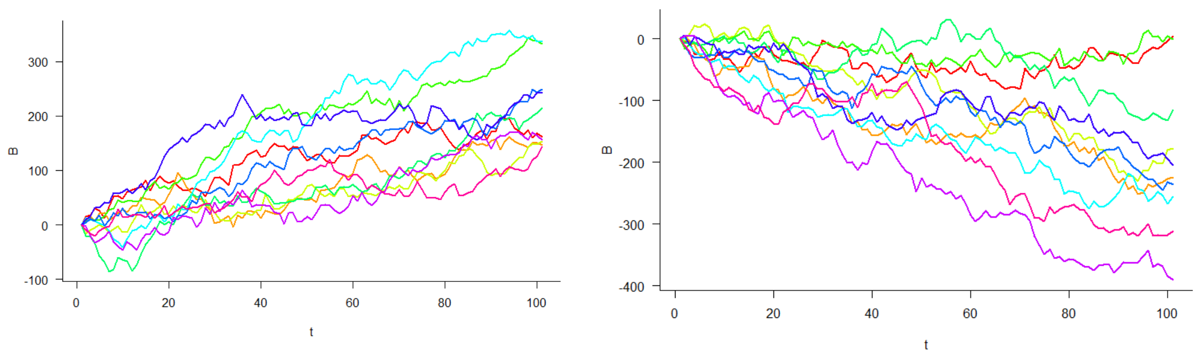

Figure 2: Brownian Motion with positve and negative drift

Commonly researchers are interested in modelling returns instead of price itself (Dhesi et al., 2016). A return is a percentage which represent the change of prices. According to Campbell, Lo, and MacKinlay (1997), there are two main reasons for using returns instead of prices themselves. The returns are scale free, and it is easier to work with returns than prices themselves.

Let S_t denotes the price at time t and μ is expected rate of return. The return or relative change in price during the time of dt can be defined as

$$dS_t = \mu S_t dt + \sigma S_t dB_t \quad (14)$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \quad (15)$$

The daily return in (14) is a stochastic differential equation (SDE) with Brownian motion with drift has two parts which one is predictable and the other one is unpredictable. $\mu S_t dt$ represents the predictable part of the return and $\sigma S_t dB_t$ represents the unexpected part. The equation in (15) defines the rate which follows an Ito's process. All intervals with the length dt, the rate can be defined as

$$\frac{dS_t}{S_t} = d(\ln S_t) = \ln S_t - \ln S_{t-1} = \mu dt + \sigma dB_t \quad (16)$$

Now there is a proposed model for the daily return and the goal is to obtain S_t. It means that to find the solution of the stochastic differential equation given by (15). To find the solution of the SDE in (15), Ito's formula is applied.

Ito's lemma:

If $\{X_t, t \geq 0\}$ is an Ito process;

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \quad (17)$$

$$X_t : Y_t = g(X_t) \quad (18)$$

$$\partial Y_t = \partial g(X_t) = \left[\frac{\partial g(X_t)}{\partial X} \mu(X_t, t) + \frac{1}{2} \frac{\partial^2 g(X_t)}{\partial X} \sigma^2(X_t, t) \right] dt + \frac{\partial g(X_t)}{\partial X} \sigma(X_t, t) dW_t \quad (19)$$

The following substitutions are inserted to the Ito's formula in (17)

$Y_t = \ln S_t$; $X_t \to S_t$; $\mu(X_t, t) = \mu S_t$; $\sigma(X_t, t) = \sigma S_t$

and the following equations are obtained.

$$d\ln S_t = \left(\frac{1}{S_t} \mu S_t + \frac{1}{2}\left(-\frac{1}{S_t^2} \right) \sigma^2 S_t^2 \right) dt + \frac{1}{S_t} \sigma S_t dW_t \tag{20}$$

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t \tag{21}$$

Recalling (10) and (16)

$$d\ln S_t = \ln S_t - \ln S_{t-1} = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma\varepsilon\sqrt{dt} \tag{22}$$

Here, $\varepsilon\sqrt{dt}$ is a random walk and $d\ln S_t$ is Brownian motion with drift and

$$d\ln S_t \sim N\left(\left(\mu - \frac{1}{2}\sigma^2 \right) dt, \sigma^2 dt \right) \tag{23}$$

$$\ln S_t = \ln S_{t-1} + \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma\varepsilon\sqrt{dt} \tag{24}$$

Finally, the solution of the SDE in (15) is obtained and it is a geometric Brownian motion model for the future stock price.

$$S_t = e^{\ln S_{t-1}} e^{\left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma\varepsilon\sqrt{dt}} \tag{25}$$

The estimations of μ and σ are calculated from the data. The daily return is calculated as follows:

$$x_i = \ln\left(\frac{S_i}{S_{i-1}} \right), \; i = 1, 2, \ldots, n \tag{26}$$

The drift and volatility of daily returns are calculated by using (26) and (27), respectively.

$$\hat{\mu} = \frac{\bar{x}}{\Delta t} + \frac{1}{2}\sigma^2 \tag{27}$$

Here, Δt represents the length of consecutive time periods.

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{28}$$

Considering the (23) , %95 confidence interval of S_t can be constructed.

$$P\left(e^{\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t - z_{0.05}\sigma\sqrt{t}} \leq S_t \leq e^{\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + z_{0.05}\sigma\sqrt{t}} \right) = 0.95 \tag{29}$$

4. APPLICATION STUDY

In this section, an historical data set which consists of daily closing prices stock from 16 March 2019-13 March 2020 of Walmart Company stock is considered and generated series of future closing prices based on the Geometric Brownian motion model in (25). The historical data set is obtained from the web site of Yahoo Finance. Considering the benefits of working with daily returns the daily returns of the data are obtained (26). Figure 3 illustrates the closing stock prices and the histogram of the daily returns.

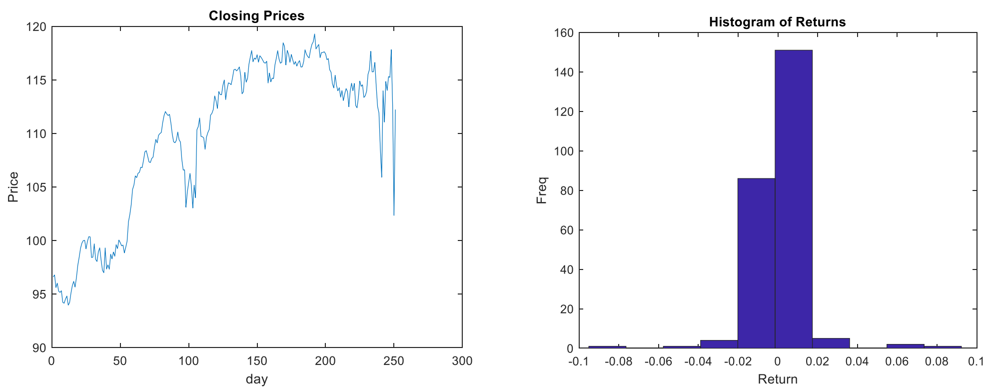

Figure 3. The graphs of closing stocks prices and daily returns.

Geometric Brownian motion is considered as an appropriate model for the prices. The drift and volatility are estimated from data and based on (25) 1000 stock prices are simulated. The average of the simulated values is calculated and %95 confidence interval of the mean of the prices, under the assumption of normality, is estimated. As an actual value the closing price of the first following day which is 16 March 2020 of the studied stock is considered and checked whether the estimated confidence interval covers the actual value. The results are listed in Table 1.

Table1: The Simulation Results

Average Simulated Expected Prices	%95 Confidence Interval	Actual Value (16.03.2020)
104.56	104.26-105.54	105.01

According to the results in Table 1, geometric Brownian motion model provides accurate predictions, which are considered to estimate confidence interval of the actual stock price.

5. CONCLUSION

Developing statistical models which provide accurate predictions for future stock prices, is studied by various researchers. In this study, geometric Brownian motion model is revisited to modelling stock prices. Basics of random walk process and the brief definition of the geometric Brownian motion are summarized. The theory behind the idea of modelling stock prices with geometric Brownian motion is discussed. A small study is conducted to present that the geometric Brownian motion is an appropriate model for stock prices, which yields accurate predictions.

ACKNOWLEDGEMENT

The author would like to thank the Chief Editor of the journal and the anonymous reviewers for their constructive comments and suggestions.

DISCLOSURE OF CONFLICT

The author declares that she has no conflicts of interest.

AUTHOR(S) DETAILS

Cigdem Topcu Guloksuz, PhD.
The Department of Management Information Systems
Ankara Bilim University, Turkey
E-mail: cigdem.topcu.guloksuz@ankarabilim.edu.tr
ORCID ID: https://orcid.org/0000-0002-7433-1297

REFERENCES

Malliaris. A. (1983). Ito's calculus in financial decision making. *SIAM Review, 25*(4), 481–496.

Mandelbrot, B. B., & van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review, 10*, 422–437. Retrieved from: Publisher Site | Google Scholar | Zentralblatt MATH | MathSciNet https://arxiv.org/ftp/arxiv/papers/1507/1507.02203.pdf. 2018-01-29.

Dhesi, G., Shakeel, N. B., & Xiao, L. (2016). *Modifed Brownian Motion Approach to Modeling Returns Distribution*. School of Business, London South Bank University.

Campbell, J.Y., Lo, A.W., & MacKinlay, A. C. (1997). *The Econometrics of Financial Markets.* Princeton University Press.

Azizaha, M. M., Irawan, & Putri, E. R. M. (2020). *Comparison of stock price prediction using geometric Brownian motion and multilayer perceptron.* AIP Conference Proceedings 2242, 030016. Retreived from: https://doi.org/10.1063/5.0008066

Almgren, R. (2002). Financial derivatives and partial differential equations. *The American Mathematical Monthly, 109*(1), 1–12.

Krishna, R., & Vaughan, C. (2016). *Simulating Stock Prices Using Geometric Brownian Motion: Evidence from Australian Companies.* Retrieved from: https://pdfs.semanticscholar.org/6778/2faf27ffeb8acd203ad3e270ee8dfb831e9e.pdf.

Rajpal, R. (2018). Mathematical Modeling in Finance. *International Journal of Scientific & Engineering Research, 9*(11).

Zhijun, Y. (2015). *Geometric Brownian Motion Model in Financial Market.* Berkeley: University of California.